Contrôle continu du 27/02/2010

Partie I

1). Comme $[\hat p_x, \hat p_y] = 0$, on a

$$[\hat p_x^2, \hat p_y^2] = \hat p_x^2 \hat p_y^2 - \hat p_y^2 \hat p_x^2 = 0$$

De même, $[\hat p_x^2, \hat y^2] = [\hat p_y^2, \hat x^2] = [\hat x^2, \hat y^2] = 0$. Alors

$$[\hat H_x, \hat H_y] = [\hat p_x^2 + \hat x^2, \hat p_y^2 + \hat y^2] = [\hat p_x^2, \hat p_y^2] + [\hat p_x^2, \hat y^2] +$$
$$+ [\hat x^2, \hat p_y^2] + [\hat x^2, \hat y^2] = 0$$

2). Par exemple,

$$[\hat H, \hat H_x] = [\hat H_x + \hat H_y, \hat H_x] = \underbrace{[\hat H_x, \hat H_x]}_{=0} + \underbrace{[\hat H_y, \hat H_x]}_{=0 \text{ d'après 1)}} = 0$$

3). $[\hat H, \hat L_z] = [\hat p_x^2 + \hat x^2 + \hat p_y^2 + \hat y^2, \hat L_z] = [\hat H_x + \hat H_y, \hat L_z] =$

$$= [\hat H_x, \hat L_z] + [\hat H_y, \hat L_z].$$

On a pourtant

$$[\hat H_x, \hat L_z] = [\hat p_x^2 + \hat x^2, \hat x \hat p_y - \hat y \hat p_x] =$$

$$= [\hat p_x^2, \hat x \hat p_y] - \underbrace{[\hat p_x^2, \hat y \hat p_x]}_{=0 \text{ car } [\hat p_x, \hat p_x] = [\hat p_x, \hat y] = 0} + \underbrace{[\hat x^2, \hat x \hat p_y]}_{=0 \text{ car } [\hat x, \hat x] = [\hat x, \hat p_y] = 0} - [\hat x^2, \hat y \hat p_x] =$$

$$= [\hat p_x^2, \hat x \hat p_y] - [\hat x^2, \hat y \hat p_x] =$$

$$= [\hat p_x^2, \hat x] \hat p_y + \hat x \underbrace{[\hat p_x^2, \hat p_y]}_{=0} - \underbrace{[\hat x^2, \hat y]}_{=0} \hat p_x - \hat y [\hat x^2, \hat p_x] =$$

$$= [\hat p_x^2, \hat x] \hat p_y - \hat y [\hat x^2, \hat p_x] =$$

$$= \hat p_x \underbrace{[\hat p_x, \hat x]}_{-i} \hat p_y + \underbrace{[\hat p_x, \hat x]}_{-i} \hat p_x \hat p_y - \hat y \underbrace{[\hat x, \hat p_x]}_{i} \hat x - \hat y \hat x \underbrace{[\hat x, \hat p_x]}_{i} =$$

$$= -i \hat p_x \hat p_y - i \hat p_x \hat p_y - i \hat y \hat x - i \hat y \hat x = -2i (\hat x \hat y + \hat p_x \hat p_y)$$

(notons que $\hat x \hat y = \hat y \hat x$ et $\hat p_x \hat p_y = \hat p_y \hat p_x$)

De la même manière,

$$[\hat{H}_y, \hat{L}_z] = [\hat{p}_y^2 + \hat{y}^2, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x] =$$

$$= \underbrace{[\hat{p}_y^2, \hat{x}\hat{p}_y]}_{=0} - [\hat{p}_y^2, \hat{y}\hat{p}_x] + [\hat{y}^2, \hat{x}\hat{p}_y] - \underbrace{[\hat{y}^2, \hat{y}\hat{p}_x]}_{=0} =$$

$$= -[\hat{p}_y^2, \hat{y}\hat{p}_x] + [\hat{y}^2, \hat{x}\hat{p}_y] =$$

$$= -[\hat{p}_y^2, \hat{y}]\hat{p}_x - \hat{y}\underbrace{[\hat{p}_y^2, \hat{p}_x]}_{=0} + \underbrace{[\hat{y}^2, \hat{x}]}_{=0}\hat{p}_y + \hat{x}[\hat{y}^2, \hat{p}_y] =$$

$$= -[\hat{p}_y^2, \hat{y}]\hat{p}_x + \hat{x}[\hat{y}^2, \hat{p}_y] =$$

$$= -\underbrace{[\hat{p}_y, \hat{y}]}_{=-i}\hat{p}_y\hat{p}_x - \hat{p}_y\underbrace{[\hat{p}_y, \hat{y}]}_{=-i}\hat{p}_x + \hat{x}\hat{y}\underbrace{[\hat{y}, \hat{p}_y]}_{=i} + \hat{x}\underbrace{[\hat{y}, \hat{p}_y]}_{=i}\hat{y} =$$

$$= 2i\,\hat{p}_x\hat{p}_y + 2i\,\hat{x}\hat{y}.$$

Donc $[\hat{H}, \hat{L}_z] = [\hat{H}_x, \hat{L}_z] + [\hat{H}_y, \hat{L}_z] = -2i(\hat{x}\hat{y} + \hat{p}_x\hat{p}_y) + 2i(\hat{x}\hat{y} + \hat{p}_x\hat{p}_y) = 0$

4). On vient de montrer que

$$[\hat{H}_y, \hat{L}_z] = 2i(\hat{x}\hat{y} + \hat{p}_x\hat{p}_y) \implies \hat{K} = \hat{x}\hat{y} + \hat{p}_x\hat{p}_y$$

5). $2i[\hat{H}, \hat{K}] = [\hat{H}, 2i\hat{K}] = [\hat{H}, [\hat{H}_y, \hat{L}_z]] =$

$$= [\hat{H}, \hat{H}_y\hat{L}_z - \hat{L}_z\hat{H}_y] =$$

$$= \hat{H}(\hat{H}_y\hat{L}_z - \hat{L}_z\hat{H}_y) - (\hat{H}_y\hat{L}_z - \hat{L}_z\hat{H}_y)\hat{H} = 0$$

(car $\hat{H}$ commute avec $\hat{H}_y$ et $\hat{L}_z$).

6). $[\hat{H}_x, \hat{K}] = [\hat{p}_x^2 + \hat{x}^2, \hat{x}\hat{y} + \hat{p}_x\hat{p}_y] =$

$$= [\hat{p}_x^2, \hat{x}\hat{y}] + \underbrace{[\hat{p}_x^2, \hat{p}_x\hat{p}_y]}_{=0} + \underbrace{[\hat{x}^2, \hat{x}\hat{y}]}_{=0} + [\hat{x}^2, \hat{p}_x\hat{p}_y]$$

$$= [\hat{p}_x^2, \hat{x}\hat{y}] + [\hat{x}^2, \hat{p}_x\hat{p}_y] =$$

$$= [\hat{p}_x^2, \hat{x}]\hat{y} + \hat{x}\underbrace{[\hat{p}_x^2, \hat{y}]}_{=0} + [\hat{x}^2, \hat{p}_x]\hat{p}_y + \hat{p}_x\underbrace{[\hat{x}^2, \hat{p}_y]}_{=0}$$

$$= [\hat{p}_x^2, \hat{x}]\hat{y} + [\hat{x}^2, \hat{p}_x]\hat{p}_y =$$

$$= \hat{p}_x \underbrace{[\hat{p}_x, \hat{x}]}_{=-i}\hat{y} + \underbrace{[\hat{p}_x, \hat{x}]}_{=-i}\hat{p}_x\hat{y} + \underbrace{[\hat{x}, \hat{p}_x]}_{=i}\hat{x}\hat{p}_y + \hat{x}\underbrace{[\hat{x}, \hat{p}_x]}_{=i}\hat{p}_y =$$

$$= -2i\,\hat{p}_x\hat{y} + 2i\,\hat{x}\,\hat{p}_y = 2i(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) = 2i\,\hat{L}_z$$

(car $\hat{p}_x\hat{y} = \hat{y}\hat{p}_x$)

Donc $[\hat{H}_x, \hat{K}] = 2i\hat{L}_z$. De plus

$$[\hat{H}_y, \hat{K}] = [\hat{H} - \hat{H}_x, \hat{K}] = \underbrace{[\hat{H}, \hat{K}]}_{\substack{= 0 \\ \text{d'après 5).}}} - [\hat{H}_x, \hat{K}] =$$

$$= -[\hat{H}_x, \hat{K}] = -2i\,\hat{L}_z.$$

Finalement :

$$[\hat{L}_z, \hat{K}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{x}\hat{y} + \hat{p}_x\hat{p}_y] =$$

$$= [\hat{x}\hat{p}_y, \hat{x}\hat{y}] + [\hat{x}\hat{p}_y, \hat{p}_x\hat{p}_y] - [\hat{y}\hat{p}_x, \hat{x}\hat{y}] - [\hat{y}\hat{p}_x, \hat{p}_x\hat{p}_y] =$$

$$= \hat{x}[\hat{p}_y, \hat{x}\hat{y}] + \underbrace{[\hat{x}, \hat{x}\hat{y}]}_{=0}\hat{p}_y + \hat{x}\underbrace{[\hat{p}_y, \hat{p}_x\hat{p}_y]}_{=0} + [\hat{x}, \hat{p}_x\hat{p}_y]\hat{p}_y$$

$$- \hat{y}[\hat{p}_x, \hat{x}\hat{y}] - \underbrace{[\hat{y}, \hat{x}\hat{y}]}_{=0}\hat{p}_x - \hat{y}\underbrace{[\hat{p}_x, \hat{p}_x\hat{p}_y]}_{=0} - [\hat{y}, \hat{p}_x\hat{p}_y]\hat{p}_x =$$

$$= \hat{x}[\hat{p}_y, \hat{x}\hat{y}] + [\hat{x}, \hat{p}_x\hat{p}_y]\hat{p}_y - \hat{y}[\hat{p}_x, \hat{x}\hat{y}] - [\hat{y}, \hat{p}_x\hat{p}_y]\hat{p}_x =$$

$$= \hat{x}\underbrace{[\hat{p}_y, \hat{x}]}_{=0}\hat{y} + \hat{x}^2\underbrace{[\hat{p}_y, \hat{y}]}_{=-i} + \underbrace{[\hat{x}, \hat{p}_x]}_{=i}\hat{p}_y^2 + \hat{p}_x\underbrace{[\hat{x}, \hat{p}_y]}_{=0}\hat{p}_y$$

$$- \hat{y}\underbrace{[\hat{p}_x, \hat{x}]}_{=-i}\hat{y} - \hat{y}\hat{x}\underbrace{[\hat{p}_x, \hat{y}]}_{=0} - \underbrace{[\hat{y}, \hat{p}_x]}_{=0}\hat{p}_y\hat{p}_x - \hat{p}_x\underbrace{[\hat{y}, \hat{p}_y]}_{=i}\hat{p}_x$$

$$= -i\,\hat{x}^2 + i\hat{p}_y^2 + i\hat{y}^2 - i\,\hat{p}_x^2 = -i(\hat{p}_x^2 + \hat{x}^2) + i(\hat{p}_y^2 + \hat{y}^2) =$$

$$= i(\hat{H}_y - \hat{H}_x).$$

7). Le spectre de $\hat{H}$ est dégénéré:

- prenons par exemple un vecteur propre commun f de $\hat{H}$ et $\hat{L}_z$, avec les valeurs propres correspondantes E et m, c'est-à-dire,

$$\hat{H} f = E f, \quad \hat{L}_z f = m f.$$

- Considérons le vecteur $g = \hat{H}_x f$. Nous avons

$$\hat{H} g = \hat{H}\hat{H}_x f = \hat{H}_x \hat{H} f = \hat{H}_x E f = E \hat{H}_x f = E g$$

$$\hat{L}_z g = \hat{L}_z \hat{H}_x f = [\hat{L}_z, \hat{H}_x] f + \hat{H}_x \hat{L}_z f =$$

$$= 2i \hat{K} f + \hat{H}_x m f =$$

$$= 2i \hat{K} f + m g$$

Donc g est un nouveau vecteur propre de $\hat{H}$ avec la même énergie (nouveau car en général $\hat{K} f \neq 0$ et donc il est impossible d'avoir $g \sim f$, sinon $\hat{L}_z g = m g$).

Partie II

$$\left(-\frac{d^2}{dx^2} + x^2 + g x^3 - E\right)\psi = 0$$

1). Tout d'abord on constate que l'ansatz naïf

$$\psi(x) \sim a x^\nu + o(x^\nu)$$

ne marche pas, car dans ce cas

$$-a\nu(\nu-1)x^{\nu-2} + o(x^{\nu-2}) + a x^{\nu+2} + o(x^{\nu+2})$$
$$+ g a x^{\nu+3} + o(x^{\nu+3}) - E a x^\nu + o(x^\nu) = 0$$

$$g a + o(1) = 0 \implies a = 0.$$

2). On essaie donc

$$\psi(x) \sim x^\nu e^{P(x)} (1 + o(1)) \quad \text{avec } P(x) = A x^d + \ldots$$

En introduisant $g = \ln \psi \iff \psi = e^g$ on a

$$\psi' = e^f f'$$

$$\psi'' = e^f (f'' + (f')^2)$$

$$f(x) = A x^d + o(x^d)$$

et alors :

$$-f'' - (f')^2 + x^2 + g x^3 - E = 0$$

$$-A\,d\,(d-1)\,x^{d-2} + o(x^{d-2}) - A^2 d^2 x^{2d-2} + o(x^{2d-2})$$
$$+ x^2 + g x^3 - E = 0$$

Nous avons 2 candidats pour le terme dominant:

- $A^2 d^2 x^{2d-2} \implies A = 0 \implies$ contradiction
- $g x^3 \implies g = 0 \implies$ contradiction

Donc ces 2 termes doivent se compenser:

$$g x^3 = A^2 d^2 x^{2d-2}$$
$$\Downarrow$$
$$\begin{cases} g = A^2 d^2 & \implies A = \pm\left(\frac{4}{25} g\right)^{1/2} = \pm \frac{2}{5}\sqrt{g} \\ 2d - 2 = 3 & \implies d = 5/2 \end{cases}$$

Comme la puissance qui apparait est <u>demi-entière</u>, il n'est pas clair a priori quelle sera la forme des termes suivants dans le développement de $f(x)$.

Au lieu d'essayer les trouver "expérimentalement", passons par changement de variables

$$x = y^2 \implies$$

$$\frac{df}{dx} = \frac{1}{2y}\frac{df}{dy}$$

$$\frac{d^2 f}{dx^2} = \frac{1}{2y}\frac{d}{dy}\frac{1}{2y}\frac{d}{dy} f = \frac{1}{4y^2}\frac{d^2 f}{dy^2} - \frac{1}{4y^3}\frac{df}{dy}$$

L'équation pour f devient

$$-\frac{1}{4y^2}\frac{d^2 f}{dy^2} + \frac{1}{4y^3}\frac{df}{dy} - \frac{1}{4y^2}\left(\frac{df}{dy}\right)^2 + y^4 + g y^6 - E = 0$$

$$-\frac{d^2f}{dy^2}+\frac{1}{y}\frac{df}{dy}-\left(\frac{df}{dy}\right)^2+4y^6+4g\,y^8-4Ey^2=0$$

On cherche la solution sous la forme

$$f(y\to\infty)\simeq A\,y^{\alpha}+o(y^{\alpha})$$

$$-A\alpha(\alpha-1)y^{\alpha-2}+o(y^{\alpha-2})+A\alpha\,y^{\alpha-2}+o(y^{\alpha-2})$$

$$-A^2\alpha^2\,y^{2\alpha-2}+o(y^{2\alpha-2})+4y^6+4g\,y^8-4Ey^2=0$$

Le raisonnement analogue au précédent conduit à

$$\begin{cases} A^2\alpha^2=4g \Rightarrow A=\pm\frac{2}{5}\sqrt{g}\\ 2\alpha-2=8 \Rightarrow \alpha=5\end{cases}$$

Le développement plus détaillé est donc de la forme

$$f(y\to\infty)\sim\pm\frac{2}{5}\sqrt{g}\;y^5+\alpha y^4+\beta y^3+\gamma y^2+\delta y+\nu\ln y$$

Donc :

$$-\underbrace{\left[\pm\frac{2}{\cancel{5}}\sqrt{g}\cdot\cancel{5}\cdot4\cdot y^3+\alpha\cdot4\cdot3\cdot y^2+\beta\cdot3\cdot2\,y+\gamma\cdot2\cdot1-\frac{\nu}{y^2}+o\left(\frac{1}{y^2}\right)\right]}_{-\frac{d^2f}{dy^2}}$$

$$+\underbrace{\frac{1}{y}\left[\pm\frac{2}{\cancel{5}}\sqrt{g}\cdot\cancel{5}\cdot y^4+\alpha\cdot4\cdot y^3+\beta\cdot3y^2+\gamma\cdot2y+\delta+\frac{\nu}{y}+o\left(\frac{1}{y}\right)\right]}_{\frac{1}{y}\frac{df}{dy}}$$

$$-\underbrace{\left[\pm\frac{2}{\cancel{5}}\sqrt{g}\cdot\cancel{5}\,y^4+\alpha\cdot4y^3+\beta\cdot3y^2+\gamma\cdot2y+\delta+\frac{\nu}{y}+o\left(\frac{1}{y}\right)\right]^2}_{\left(\frac{df}{dy}\right)^2}+$$

$$+\;4y^6+4g\,y^8-4Ey^2\quad=0$$

Le terme inconnu le plus grand correspond au produit

$$\pm\frac{2}{5}\sqrt{g}\cdot5y^4\cdot o\left(\frac{1}{y}\right),\qquad(\sim o(y^3))$$

donc on gardera les termes jusqu'à y^3 dans les développements.

$$\mp \frac{2}{\cancel{5}}\sqrt{g}\cdot \cancel{5}\cdot 4\cdot y^3 \pm \frac{2}{\cancel{5}}\sqrt{g}\cdot\cancel{5}\, y^3$$

$$-4\cancel{g\,y^8} - 16\cancel{\alpha^2 y^6} - 9\beta^2 y^4$$

$$\mp 2\cdot 2\cancel{\sqrt{g}\cdot 4\alpha\, y^7} \mp 2\cdot 2\sqrt{g}\cdot 3\beta\, y^6 \mp 2\cdot 2\sqrt{g}\cdot 2\gamma\, y^5$$

$$\mp 2\cdot 2\sqrt{g}\cdot \delta\, y^4 \mp 2\cdot 2\sqrt{g}\cdot \nu\, y^3$$

$$-2\cdot 4\cancel{\alpha\cdot 3\beta}\cdot y^5 - 2\cdot 4\cancel{\alpha\cdot 2\gamma}\, y^4 - 2\cdot 4\cancel{\alpha\cdot \delta}\, y^3$$

$$-2\cdot 3\beta\cdot 2\gamma\cdot y^3 + 4y^6 + 4\cancel{g\,y^8} - 4\underbrace{\xi\, y^2}_{o(y^3)} + o(y^3) = 0$$

On voit que $\alpha = 0$ (coef. de y^7)

Donc

$$\mp 6\sqrt{g}\, y^3 - 9\beta^2 y^4 \mp 2\cdot 2\sqrt{g}\cdot 3\beta\cdot y^6 \mp 2\cdot 2\cancel{\sqrt{g}\cdot 2\gamma}\, y^5$$

$$\mp 2\cdot 2\sqrt{g}\cdot \delta\, y^4 \mp 2\cdot 2\sqrt{g}\cdot \nu\, y^3 - 2\cdot 3\cancel{\beta\cdot 2\gamma}\cdot y^3$$

$$+4y^6 + o(y^3) = 0$$

Par la suite $\gamma = 0$ (grâce au coef. de y^5), alors

$$(4 \mp 12\sqrt{g}\,\beta)\,y^6 + (-9\beta^2 \mp 4\sqrt{g}\,\delta)\,y^4$$

$$+ (\mp 6\sqrt{g} \mp 4\sqrt{g}\,\nu)\,y^3 + o(y^3) = 0.$$

Finalement :

$$\beta = \pm \frac{1}{3\sqrt{g}} \qquad \text{(coef. de } y^6)$$

$$\delta = \mp \frac{9\beta^2}{4\sqrt{g}} = \mp \frac{1}{4g\sqrt{g}} \qquad \text{(coef. de } y^4)$$

$$\nu = -\frac{3}{2} \qquad \text{(coef. de } y^3)$$

$$S(y\to\infty) \sim \pm\left[\frac{2}{5}\sqrt{g}\, y^5 + \frac{1}{3\sqrt{g}}\, y^3 - \frac{1}{4g\sqrt{g}}\, y\right] - \frac{3}{2}\ln y$$

$$\Downarrow$$

$$\psi(y\to\infty) = y^{-3/2}\, e^{\pm\left[\frac{2}{5}\sqrt{g}\, y^5 + \frac{1}{3\sqrt{g}}\, y^3 - \frac{1}{4g\sqrt{g}}\, y\right]}\,(1+o(1))$$

En termes de x :

$$\psi(x \to \pm\infty) = |x|^{-3/4}\, e^{\pm\left[\frac{2}{5}\sqrt{g}|x|^{5/2} + \frac{1}{3\sqrt{g}}|x|^{3/2} - \frac{1}{4g\sqrt{g}}|x|^{1/2}\right]}(1+o(1)).$$

Partie III

Les valeurs propres de $\hat{L}^2$ sont de la forme $\ell(\ell+1)$, où ℓ note le nombre quantique orbital.

$$\ell(\ell+1) = 6 \Rightarrow \ell^2 + \ell - 6 = 0$$

$$\ell_1 = 2, \quad \ell_2 = -3$$

$\uparrow$ cette solution est exclue car $\ell = 0, 1, 2 \ldots$ (≥ 0)

Donc

1). les valeurs propres possibles de L_z sont

$-2, -1, 0, 1, 2$ (c'est-à-dire, $-\ell, -\ell+1, \ldots, \ell$).

2). les fonctions propres communes orthonormées sont $Y_2^{-2}(\theta,\varphi)$, $Y_2^{-1}(\theta,\varphi)$, $Y_2^{0}(\theta,\varphi)$, $Y_2^{1}(\theta,\varphi)$, $Y_2^{2}(\theta,\varphi)$

explicitement :

$$Y_\ell^m(\theta,\varphi) = C_\ell^m\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

Notons que $C_\ell^m = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}$ et donc

$$C_2^2 = \sqrt{\frac{5}{4\pi}\cdot\frac{(2-2)!}{(2+2)!}} = \sqrt{\frac{5}{96\pi}}$$

$$C_2^1 = \sqrt{\frac{5}{4\pi}\frac{(2-1)!}{(2+1)!}} = \sqrt{\frac{5}{24\pi}}$$

$$C_2^0 = \sqrt{\frac{5}{4\pi}}$$

De plus

$$P_\ell^m(\cos\theta) = \frac{(\sin\theta)^m}{2^\ell \ell!}\left[\frac{d^{\ell+m}}{dz^{\ell+m}}(z^2-1)^\ell\right]_{z=\cos\theta} =$$

$$= (-1)^m \frac{(\ell+m)!}{(\ell-m)!} P_\ell^{-m}(z)\Big|_{z=\cos\theta} =$$

$$= (-1)^m \frac{(\ell+m)!}{(\ell-m)!}\cdot\frac{(\sin\theta)^{-m}}{2^\ell \ell!}\left[\frac{d^{\ell-m}}{dz^{\ell-m}}(z^2-1)^\ell\right]_{z=\cos\theta}$$

et donc

$$P_2^2(\cos\theta) = \frac{(2+2)!}{(2-2)!} \cdot \frac{(\sin\theta)^{-2}}{2^2\cdot 2!} \left[\frac{d^0}{dz^0}(z^2-1)^2\right]_{z=\cos\theta} =$$

$$= \frac{4!}{4\cdot 2} \; \frac{1}{\sin^2\theta} \; \underbrace{(\cos^2\theta - 1)}_{-\sin^2\theta}{}^2 = 3\sin^2\theta$$

$$P_2^1(\cos\theta) = (-1)\frac{(2+1)!}{(2-1)!} \; \frac{(\sin\theta)^{-1}}{2^2\cdot 2!} \left[\frac{d^{2-1}}{dz^{2-1}}(z^2-1)^2\right]_{z=\cos\theta} =$$

$$= -\frac{3}{4}\cdot\frac{1}{\sin\theta}\cdot\left[\frac{d}{dz}(z^2-1)^2\right]_{z=\cos\theta} =$$

$$= -\frac{3}{4}\cdot\frac{1}{\sin\theta}\cdot\left[2(z^2-1)\cdot 2z\right]_{z=\cos\theta} =$$

$$= -\frac{3}{4}\cdot\frac{1}{\sin\theta}\cdot 4\cos\theta\,\underbrace{(\cos^2\theta-1)}_{-\sin^2\theta} = 3\sin\theta\cos\theta$$

$$P_2^0(\cos\theta) = \frac{1}{2^2\cdot 2!}\left[\frac{d^2}{dz^2}(z^2-1)^2\right]_{z=\cos\theta} =$$

$$= \frac{1}{8}\left[\frac{d^2}{dz^2}\left(z^4-2z^2+1\right)\right]_{z=\cos\theta} =$$

$$= \frac{1}{8}\left(12z^2-4\right)_{z=\cos\theta} = \frac{3\cos^2\theta-1}{2}$$

Par conséquent, nous avons

$$Y_2^2(\theta,\varphi) = \sqrt{\frac{5}{96\pi}}\cdot 3\sin^2\theta\, e^{2i\varphi} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\varphi}$$

$$Y_2^1(\theta,\varphi) = \sqrt{\frac{5}{24\pi}}\cdot 3\sin\theta\cos\theta\, e^{i\varphi} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\varphi}$$

$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{4\pi}}\cdot\frac{3\cos^2\theta-1}{2}$$

De plus, en utilisant $Y_\ell^{-m}(\theta,\varphi) = (-1)^m\,\overline{Y_\ell^m(\theta,\varphi)}$, on a

$$Y_2^{-2}(\theta,\varphi) = \overline{Y_2^2(\theta,\varphi)} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\varphi}$$

$$Y_2^{-1}(\theta,\varphi) = -\overline{Y_2^1(\theta,\varphi)} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\varphi}.$$